Functional Analysis

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Lecture 1

Banach spaces

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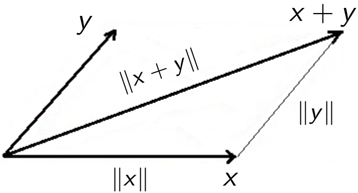
Normed spaces

What is a vector space (a linear space)?

Def. A **normed space** is a linear space X over a field $\mathbb{F} := \mathbb{R}, \mathbb{C}$ equipped with a **norm**, i.e. a function $\|\cdot\| : X \to [0, \infty)$ such that for $x, y \in X$, $\lambda \in \mathbb{F}$ we have:

- N1 $\|x\| = 0 \iff x = 0,$ (*nondegeneracy*)
- N2 $\|\lambda x\| = |\lambda| \cdot \|x\|,$ (*positive homogeneity*)
- N3 $\|x + y\| \leqslant \|x\| + \|y\|.$ (*triangle inequality*)



(*inverse triangle inequality*)

$$\big|\|x\| - \|y\|\big| \leqslant \|x - y\|$$

The norm defines a **metric** by the formula $d(x, y) := \|x - y\|$, $x, y \in X$. Hence the linear space X is also a **topological space**.

Open sets in X are unions of open balls, where by an **open ball** centered in $x \in X$ and radius $r > 0$ we mean the set

$$B(x,r) := \{y \in X : \|x - y\| < r\}.$$

A sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ **converges** to $x \in X \iff \lim_{n\to\infty} \|x_n - x\| = 0$.

We write then $x_n \to x$ or $x_n \xrightarrow{\|\cdot\|} x$.

Prop. (continuity of the linear structure) In every normed space, scalar multiplication, vector addition and norm are continuous functions.

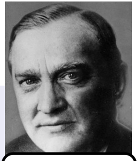
Proof: Let $\lambda_n \to \lambda$, $x_n \to x$ and $y_n \to y$, that is $|\lambda_n - \lambda| \to 0$, $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$. Then

$$\begin{aligned}\|\lambda_n x_m - \lambda x\| &\overset{N3}{\leqslant} \|\lambda_n x_m - \lambda_n x\| + \|\lambda_n x - \lambda x\| \\ &\overset{N2}{=} |\lambda_n| \cdot \|x_m - x\| + |\lambda_n - \lambda| \cdot \|x\| \longrightarrow 0, \text{ when } n, m \to \infty.\end{aligned}$$

Hence $\lambda_n x_m \to \lambda x$, and so scalar multiplication $\cdot : \mathbb{F} \times X \to X$ is continuous. Continuity of vector addition $+ : X \times X \to X$ follows from

$$\|(x_n + y_m) - (x + y)\| \overset{N3}{\leqslant} \|x_n - x\| + \|y_m - y\| \longrightarrow 0, \quad \text{when } n, m \to \infty$$

Continuity of the norm follows from the "inverse triangle inequality": $\big|\|x_n\| - \|x\|\big| \leqslant \|x_n - x\| \to 0$, whence $\|x_n\| \to \|x\|$. ■



Stefan Banach

Def. A **Banach space** is a complete normed space, that is a normed space $(X, \|\cdot\|)$, where for every $\{x_n\}_{n=1}^{\infty} \subseteq X$

$$\underbrace{\lim_{n,m\to\infty} \|x_n - x_m\| = 0}_{\text{Cauchy sequence}} \implies \underbrace{\exists_{x_0\in X} \lim_{n\to\infty} \|x_n - x_0\| = 0.}_{\text{convegent sequence}}$$

Remark. The converse implication always holds.

Ex. The real numbers $\mathbb{R}$ with norm $\|x\| = |x|$ form a Banach space over $\mathbb{R}$. Similarly, $\mathbb{C}$ with $\|x\| = |x|$ is a Banach space over $\mathbb{C}$.

Ex. The n-dimensional space $\mathbb{F}^n$ with norm $\|x\|_2 := \sqrt{\sum_{k=1}^{n} |x(k)|^2}$ is a Banach space over $\mathbb{F} = \mathbb{R}, \mathbb{C}$. (*Euclidean space*)

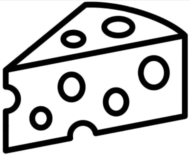
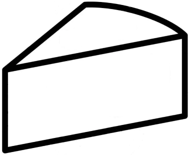
Prop. If a linear subspace $Y \subseteq X$ of a normed space X is complete, then Y is closed in X. If X is complete, then

$$Y \text{ is complete} \iff Y \text{ is closed.}$$

(recall that $\overline{Y} = Y \cup Y^d$, where Y^d is the set of limit points of Y)

Dowód: „$\Longrightarrow$". Let $y \in Y^d$, so there is $\{y_n\}_{n=1}^{\infty} \subseteq Y$ convergent to $y \in X$. In particular, $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence in Y. Hence by completeness of Y, the sequence $\{y_n\}_{n=1}^{\infty}$ converges in Y. Thus $y \in Y$. We showed that $Y^d \subseteq Y$. Equivalently, $Y = \overline{Y}$ is closed.

„$\Longleftarrow$". Assume X is complete. Any Cauchy sequence $\{y_n\}_{n=1}^{\infty} \subseteq Y$ in Y is also Cauchy in X. Hence it converges in X. Thus there is $y \in X$ such that $y_n \xrightarrow{\|\cdot\|} y$. Since $\{y_n\}_{n=1}^{\infty} \subseteq Y$ and $Y = \overline{Y}$ we must have $y \in Y$. Hence $\{y_n\}_{n=1}^{\infty}$ converges in Y. Accordingly, Y is complete. ■

$Y =$  $\Longrightarrow$ $\overline{Y} =$ 

Rem. Every normed space can be completed to a Banach space (in an essentially unique way)

Theorem (Completion of normed spaces)

For any normed space Y there exist

1. a Banach space X (the **completion** of Y)
2. linear isometry $\Psi : Y \to X$ such that $\overline{\Psi(Y)} = X$.

That is, Y embeds into X as a dense subspace. Moreover, such X is unique up to a canonical isometric isomorphism. One writes $X = \overline{Y}$.

„Sketch of Proof”: Define elements of X as equivalence classes of Cauchy sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \subseteq Y$ with respect to the equivalence relation:

$$\{x_n\} \sim \{y_n\} \overset{def}{\iff} \lim_{n\to\infty} \|x_n - y_n\| = 0.$$

Denoting by $[\{x_n\}]$ the corresponding equivalence class put

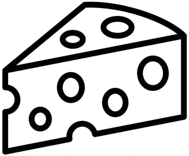
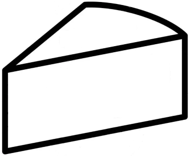
$$[\{x_n\}] + [\{y_n\}] := [\{x_n + y_n\}], \qquad \lambda[\{x_n\}] := [\{\lambda x_n\}],$$

$$\|[\{x_n\}]\| := \lim_{n\to\infty} \|x_n\|, \qquad \Psi(y) = [\{y, y, y, \ldots\}]$$

One checks all this works.

Cor. A normed space Y is complete if and only if its image $\Psi(Y)$ is closed under any linear isometry $\Psi : Y \to X$ into a normed space X.

„Completeness” = „closedness in every supspace”

$Y =$  vs $\overline{Y} =$ 

Proof: Note that for any linear isometry $\Psi : Y \to X$ the space Y is complete $\iff$ the space $\Psi(Y)$ is complete.

- If this holds, then the set $\Psi(Y)$ is closed in Y by **Prop**.
- Conversely, by **Theorem** there is an isometry $\Psi : Y \to X$ into the complete space X. Therefore, if $\Psi(Y)$ is closed in X, then by **Prop** the space $\Psi(Y) \cong Y$ is complete. ■

Def. Banach subspace $\equiv$ a closed linear subspace of a Banach space

Lem. Let $\{x_n\}_{n=1}^{\infty} \subseteq X$ be a Cauchy sequence.

1. If a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x.
2. For every $q \in (0,1)$ there is a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\|x_{n_k} - x_{n_l}\| \leqslant q^k$ for every $l \geqslant k$.

Proof: 

Def. For a sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ in a normed space X we say that **the series** $\sum\limits_{n=1}^{\infty} x_n$ **converges**, if the sequence $\left\{ \sum\limits_{n=1}^{N} x_n \right\}_{N=1}^{\infty}$ of partial sums converges and then **the sum of the series** is the limit

$$\sum_{n=1}^{\infty} x_n := \lim_{N\to\infty} \sum_{n=1}^{N} x_n$$

The series $\sum\limits_{n=1}^{\infty} x_n$ is **absolutly convergent** if $\sum\limits_{n=1}^{\infty} \|x_n\| < \infty$.

Prop. A normed space X is complete if and only if every absolutely convergent series is convergent in X.

Proof: Assume that $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Then for $M \geqslant N \geqslant 1$ we have

$$\|\sum_{n=1}^{M} x_n - \sum_{n=1}^{N} x_n\| = \|\sum_{n=N}^{M} x_n\| \leqslant \sum_{n=N}^{M} \|x_n\| \longrightarrow 0, \quad \text{when } N, M \to \infty,$$

Hence $\left\{ \sum_{n=1}^{N} x_n \right\}_{N=1}^{\infty}$ is Cauchy, and so convergent if X is complete.

Now assume that every absolutely convergent series is convergent. Let $\{x_n\}_{n=1}^{\infty} \subseteq X$ be any Cauchy sequence. By **Lem** we may choose a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that $\|x_{n_k} - x_{n_l}\| \leqslant (1/2)^k$ for $l \geqslant k$. Put $y_1 := x_{n_1}$ and $y_{k+1} := x_{n_{k+1}} - x_{n_k}$ dla $k \geqslant 1$. Then

$$\sum_{k=1}^{\infty} \|y_k\| = \|x_{n_1}\| + \sum_{k=1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| \leqslant \|x_{n_1}\| + \sum_{k=1}^{\infty} (1/2)^k = \|x_{n_1}\| + 1 < \infty$$

Hence $\sum_{k=1}^{\infty} y_k$ converges. But $\sum_{k=1}^{N} y_k = x_{n_N}$, and so in fact the subsequnce $\{x_{n_k}\}_{k=1}^{\infty}$ converges. Hence $\{x_n\}_{n=1}^{\infty}$ converges by **Lem**. ■